

# CONRAD'S PARTIAL ORDER ON P.Q.-BAER \*-RINGS

Anil Khairnar

Department of Mathematics, Abasaheb Garware College, Pune-411004, India.

*anil.khairnar@mesagc.org; anil\_maths2004@yahoo.com*

B. N. Waphare

Center for Advanced Studies in Mathematics, Department of Mathematics,

Savitribai Phule Pune University, Pune-411007, India.

*bnwaph@math.unipune.ac.in; waphare@yahoo.com*

**ABSTRACT.** We prove that a p.q.-Baer \*-ring forms a pseudo lattice with Conrad's partial order and also characterize p.q.-Baer \*-rings which are lattices. The initial segments of a p.q.-Baer \*-ring with Conrad's partial order are shown to be orthomodular posets.

**Keywords:** Conrad's partial order, p.q.-Baer \*-ring, central cover, orthomodular set.

## 1. INTRODUCTION

A *\*-ring*  $R$  is a ring equipped with an involution  $x \rightarrow x^*$ , that is an additive anti-automorphism of period at most two. An element  $e$  of a \*-ring  $R$  is a *projection* if  $e = e^2$  (idempotent) and  $e = e^*$  (self adjoint). For a nonempty subset  $B$  of  $R$ , we write  $r_R(B) = \{x \in R \mid bx = 0, \text{ for every } b \in B\}$ , and call the *right annihilator* of  $B$  in  $R$ . Similarly, we define the *left annihilator* of  $B$  in  $R$  (denoted by  $l_R(B)$ ). A ring is said to be abelian if its every idempotent is central. A ring without nonzero nilpotent elements is called a reduced ring. Let  $P$  be a poset and  $a, b \in P$ , then the join of  $a$  and  $b$ , denoted by  $a \vee b$  is defined as  $a \vee b = \sup \{a, b\}$  and the meet of  $a$  and  $b$ , denoted by  $a \wedge b$  is defined as  $a \wedge b = \inf \{a, b\}$ . A poset  $P$  is said to be a pseudo lattice, if for  $a, b \in P$ , whenever  $a, b$  have a common upper bound, then  $a \wedge b$  and  $a \vee b$  both exist.

Kaplansky [15] introduced Baer rings and Baer \*-rings to abstract various properties of  $AW^*$  algebras, von Neumann algebras and complete \*-regular rings. The concept of a Baer \*-ring is naturally motivated in the study of functional analysis. Early motivation for studying rings with involution came from rings of operators.

The set of projections in a Rickart \*-ring  $R$  forms an orthomodular lattice under the partial order ' $e \leq_p f$  if and only if  $e = fe = ef$ '. This lattice is extensively studied in [3, 15, 18]. In [2, 9, 10, 12, 19] the authors studied partial orders on complex matrices or  $\mathcal{B}(H)$  (the algebra of all bounded linear operators on an infinite-dimensional Hilbert space  $H$ ). In [11, 14, 17] the authors studied partial orders on Rickart \*-rings. Hartwig [12] defined the plus partial order on the set of regular elements in a semigroup. For  $m \times n$  matrices over a division ring  $D$  (that is  $D_{m \times n}$ ). Hartwig [12] use the concept of rank  $\rho(\cdot)$  and obtained the following result, which characterized the plus order for the ring  $D_{m \times n}$ .

**Theorem 1.1** (Theorem 2, [12]). *Let  $A, B \in D_{m \times n}$ . Then  $A \leq B$  if and only if  $\rho(B - A) = \rho(B) - \rho(A)$ . In particular, rank-subtractivity is a partial-ordering relation on  $D_{m \times n}$ .*

Also in the same paper [12], Hartwig posed the following open problems.

**Problem 1:** Can one induce a partial ordering on a ring  $R$ , by a subtractive rank-like function  $\rho : R \rightarrow G$ , where  $G$  is a well-ordered abelian group and  $\rho(b - a) = \rho(b) - \rho(a)$ ?

**Problem 2:** Does  $a \leq c$ ,  $b \leq c$ ,  $aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq c$ ?

Conrad [8] extended the work of Abian [1] by showing that a ring  $R$  is partially ordered by the relation  $a \leq_c b$  if and only if  $arb = ara$  for all  $r \in R$  (this is called Conrad's relation) precisely when it is semiprime. Burgess and Raphael [6] proved that this relation, when defined on a semigroup  $S$ , is a partial order whenever  $S$  is weakly separative.

Birkenmeier et al. [4] introduced principally quasi-Baer (p.q.-Baer) \*-rings. A \*-ring  $R$  is said to be a *p.q.-Baer \*-ring* if, for every principal right ideal  $aR$  of  $R$ ,  $r_R(aR) = eR$ , where  $e$  is a projection in  $R$ . From the above definition, it follows that  $l_R(aR) = Rf$  for a suitable projection  $f$ . There is an abelian p.q.-Baer \*-ring which is not a Rickart \*-ring. Also, reduced Rickart \*-rings are p.q.-Baer \*-rings. In [4], Birkenmeier et al. have given examples p.q.-Baer \*-rings those are not Rickart \*-rings or quasi-Baer \*-rings. It is easy to observe that p.q.-Baer \*-rings are semiprime. Therefore Conrad's relation is a partial order on a p.q.-Baer \*-ring.

Let  $R$  be a \*-ring and  $x \in R$ , we say that  $x$  possesses a *central cover* if there exists a smallest central projection  $h$  such that  $hx = x$ . If such a projection  $h$  exists, then it is unique, and is called the central cover of  $x$ , denoted by  $h = C(x)$  (see [3]). In [16] the authors proved the existence of central cover of every element of a p.q.-Baer \*-ring. In the second section of this paper, we characterize Conrad's partial order on p.q.-Baer \*-rings in terms of central covers. Also, we prove the result similar to Theorem 1.1, in connection to Problem 1.

In [5], Blackwood et al. answered Problem 2 negatively for the minus partial order on the ring of matrices over a field. In the third section, we answer Problem 2 positively, for p.q.-Baer \*-rings with Conrad's partial order.

Janowitz [14] proved that the initial segments of an arbitrary Rickart \*-ring with the \*-order are orthomodular posets. The same result is proved by Kr emere [17] for the left-star order. In the last section, we prove that the initial segments of a p.q.-Baer \*-ring with Conrad's partial order are orthomodular posets.

## 2. CONRAD'S RELATION ON P.Q.-BAER \*-RINGS

In this section, we characterize Conrad's partial order on a p.q.-Baer \*-ring in terms of central covers of elements. Also, we construct a subtractive function in terms of central covers, which induces Conrad's partial order on a p.q.-Baer \*-ring.

**Remark 2.1.** Let  $R$  be a \*-ring and  $P(Z(R))$  denotes the set of central projections of  $R$ .

- (1) If we restrict the Conrad's relation to the set  $P(Z(R))$ , then the relation becomes a partial order on  $P(Z(R))$ . Further, for  $e, f \in P(Z(R))$ ,  $e \leq f$  if and only if  $e = ef$ .
- (2) For any  $e \in P(Z(R))$  the central cover  $C(e)$  exists and  $C(e) = e$ . Moreover, whenever  $C(x)$  exists for some  $x \in R$ , then for any  $e \in P(Z(R))$ , the central cover  $C(ex)$  exists and  $C(ex) = eC(x)$ .
- (3) Let  $a \in R$ . If  $C(a)$  exists in  $R$ , then  $C(a^*)$  exists in  $R$  and  $C(a^*) = C(a)$  (see [16]).

Hence fourth,  $\leq$  denotes the Conrad's partial order on p.q.-Baer \*-ring.

**Lemma 2.2.** *Let  $R$  be a \*-ring and  $x \in R$ . Let  $e \in R$  be a central projection in  $R$  such that (1)  $xe = x$  and (2)  $xRy = 0$  implies  $ey = 0$ . Then  $e = C(x)$ .*

*Proof.* To prove that  $e = C(x)$ , it is sufficient to prove that  $e$  is the smallest central projection with  $xe = x$ . Let  $e' \in R$  be a central projection such that  $xe' = x$ . Then  $x(1 - e') = 0$ . Since  $1 - e'$  is central,  $xR(1 - e') = 0$ . By condition (2), we have  $e(1 - e') = 0$  and hence  $e = ee'$ . Therefore  $e \leq e'$ . Thus  $e = C(x)$ .  $\square$

The following result [16], gives the existence of central cover of every element in a p.q.-Baer \*-ring.

**Theorem 2.3** (Theorem 2.3, [16]). *Let  $R$  be a p.q.-Baer \*-ring and  $x \in R$ . Then  $x$  has a central cover  $e \in R$ . Further,  $xRy = 0$  if and only if  $yRx = 0$  if and only if  $ey = 0$ . That is  $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1 - e)R = R(1 - e)$ .*

In the following theorem we characterize the Conrad's relation in terms of central cover.

**Lemma 2.4.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . Then the following statements are equivalent.*

- (1)  $a^*rb = a^*ra$  for all  $r \in R$ .
- (2)  $a = C(a)b$ .
- (3)  $arb = ara$  for all  $r \in R$  (that is  $a \leq b$ ).

*Proof.* (1)  $\Rightarrow$  (2): By (1),  $a^*r(b - a) = 0$  for all  $r \in R$ , that is  $a^*R(b - a) = 0$ . By Theorem 2.3, we get  $C(a^*)(b - a) = 0$ . This implies  $C(a)(b - a) = 0$  (by Remark 2.1). Consequently,  $a = C(a)b$ .

(2)  $\Rightarrow$  (3): For  $r \in R$ , we have by (2),  $ara = arC(a)b = C(a)arb = arb$ . Therefore  $arb = ara$  for all  $r \in R$ .

(3)  $\Rightarrow$  (1): By the similar arguments as in the proof of (1)  $\Rightarrow$  (2), we get  $a = C(a)b$ . Further, for  $r \in R$ ,  $a^*ra = a^*rC(a)b = C(a)a^*rb = C(a^*)a^*rb = a^*rb$ . Thus  $a^*rb = a^*ra$  for all  $r \in R$ .  $\square$

The above lemma essentially says that, in a p.q.-Baer \*-ring  $R$ , for  $a, b \in R$ ,  $a \leq b$  if and only if  $a = C(a)b$ . Therefore, we use the relation  $a = C(a)b$  as Conrad's relation (partial order) on a p.q.-Baer \*-ring. The following lemma leads to the result which constructs a subtractive function on a p.q.-Baer \*-ring.

**Lemma 2.5.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  be such that  $a \leq b$ . Then,*

- (1)  $C(a) \leq C(b)$  and  $a = aC(b) = bC(a)$
- (2)  $C(b - a) = C(b) - C(a)$ .

*Proof.* (1): As  $a \leq b$ ,  $a = C(a)b$  and hence  $C(a) = C(C(a)b) = C(a)C(b)$  (by Remark 2.1). This yields  $C(a) \leq C(b)$ . By multiplying by  $a$  to both sides of  $C(a) = C(a)C(b)$  we get,  $a = aC(b)$ . Therefore  $a = aC(b) = bC(a)$ .

(2): As,  $C(a) \leq C(b)$ ,  $C(b) - C(a)$  is a central projection with  $(b - a)(C(b) - C(a)) = bC(b) - bC(a) - aC(b) + aC(a) = b - a - a + a = b - a$  (by part (1)). Further for  $y \in R$ ,  $(b - a)Ry = 0$  if and only if  $bry = ary$  for all  $r \in R$  if and only if  $bC(b)ry = bC(a)ry$  for all  $r \in R$  if and only if  $bR(C(b) - C(a))y = 0$  if and only if  $C(b)(C(b) - C(a))y = 0$  (by Theorem 2.3) if and only if  $(C(b) - C(a))y = 0$ . Thus, by Lemma 2.2,  $C(b - a) = C(b) - C(a)$ , as required.  $\square$

In the above lemma we have proved that in a p.q.-Baer \*-ring  $R$ , for  $a, b \in R$ , if  $a \leq b$  then  $C(b - a) = C(b) - C(a)$ . The following lemma gives a sufficient condition so that the converse of this statement is true.

**Lemma 2.6.** *Let  $R$  be a p.q.-Baer \*-ring in which 2 is invertible. Let  $a, b \in R$  be such that  $C(b - a) = C(b) - C(a)$ . Then  $a \leq b$ .*

*Proof.* Let  $a, b \in R$  be such that  $C(b - a) = C(b) - C(a)$ . Then  $(C(b) - C(a))^2 = C(b) - C(a)$ , which yields  $2C(b)C(a) = 2C(a)$ . Since 2 is invertible element in  $R$ , we have  $C(b)C(a) = C(a)$ . Further,  $C(b - a)C(a) = (C(b) - C(a))C(a) = 0$ . By Theorem 2.3,  $(b - a)RC(a) = 0$ . Consequently,  $(b - a)C(a) = 0$  and hence  $bC(a) = a$ . Therefore  $a \leq b$ .  $\square$

The following theorem characterises Conrad's partial order in terms of central covers, which gives a result similar to Theorem 1.1.

**Theorem 2.7.** *Let  $R$  be a p.q.-Baer \*-ring in which 2 is invertible and let  $a, b \in R$ . Then  $a \leq b$  if and only if  $C(b - a) = C(b) - C(a)$ .*

*Proof.* The proof follows from Lemmas 2.5 and 2.6.  $\square$

### 3. WHEN DOES A P.Q.-BAER \*-RING BECOME A LATTICE?

Hartwig [13] showed that a star-regular ring  $R$  forms a pseudo upper semilattice under the star-orthogonal partial ordering. That is, for every  $a, b \in R$ ,  $a, b$  have a common upper bound if and only if  $a \vee b$  exists in  $R$ . In this section, we prove that, in a p.q.-Baer \*-ring  $R$  with Conrad's partial order, for every  $a, b \in R$ ,  $a, b$  have a common upper bound if and only if  $a \vee b$  exists in  $R$ . Also, we give a sufficient condition for p.q.-Baer \*-ring to be a lattice. As a consequence, we answer Problem 2 positively for p.q.-Baer \*-rings with Conrad's partial order.

**Definition 3.1.** In [8], a concept of orthogonality is introduced as follows. Let  $R$  be a semiprime ring and  $a, b \in R$ . Then  $a$  is said to be *orthogonal* to  $b$  if  $aRb = 0$ . In a p.q.-Baer \*-ring this condition is equivalent to  $C(a)C(b) = 0$  (see [16]). We write  $a \perp b$ , whenever  $a$  is orthogonal to  $b$ .

Recall the following definition and theorem from [6].

**Definition 3.2.** Let  $R$  be a semiprime ring. For an ideal  $I$  of  $R$ ,  $\text{Ann } I = \{r \in R \mid rI = 0\}$ . If for each ideal  $I$ ,  $\text{Ann } I$  contains a nonzero central idempotent then  $R$  is called *weakly  $i$ -dense*.  $R$  is *orthogonally complete* if every orthogonal set has a supremum.

**Theorem 3.3** (Theorem 9, [6]). *An orthogonally complete semiprime ring which is weakly  $i$ -dense is complete.*

We give an example of a commutative, reduced, weakly  $i$ -dense p.q.-Baer \*-ring which is not orthogonally complete.

**Example 3.4.** Let  $R = \{x \in \prod_{i=1}^{\infty} Q \mid \text{for almost all } i, x_i \in \mathbb{Z}\}$ . Then  $R$  is a commutative \*-ring with an identity involution. For  $a = (a_1, a_2, \dots) \in R$ ,  $r_R(a) = bR$  where  $b = (b_1, b_2, \dots)$  with  $b_i = 1$  if  $a_i = 0$ ; and  $b_i = 0$  if  $a_i \neq 0$ . Note that  $b^2 = b = b^*$ . Therefore  $R$  is a Rickart \*-ring. Since an abelian Rickart \*-ring is a reduced p.q.-Baer \*-ring,  $R$  becomes a commutative reduced p.q.-Baer \*-ring. Since every ideal of  $R$  is a principal

ideal and  $R$  is a p.q.-Baer \*-ring, therefore by Theorem 2.3,  $R$  is weakly  $i$ -dense. Let  $c_1 = (\frac{1}{2}, 0, 0, \dots)$ ,  $c_2 = (0, \frac{1}{2}, 0, 0, \dots)$ ,  $\dots$ , and  $S = \{c_n \mid n \in \mathbb{N}\}$ . Then  $S$  is an orthogonal subset of  $R$  which does not have supremum in  $R$ . Thus  $R$  is not orthogonally complete.

In the following theorem, we prove that a p.q.-Baer \*-ring forms a pseudo lattice with respect to Conrad's partial order.

**Theorem 3.5.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  have a common upper bound. Then*

- (1)  $aC(b) = bC(a)$ ;
- (2)  $a^*rb = C(a)b^*rb = C(b)a^*ra$  for all  $r \in R$ . Hence,  $a^*b$  is self adjoint;
- (3)  $arb^* = C(a)brb^* = C(b)ara^*$  for all  $r \in R$ . Hence,  $ab^*$  is self adjoint;
- (4)  $a \wedge b = aC(b) = bC(a)$ ; and
- (5)  $a \vee b = a + b - a \wedge b$ .

*Proof.* Let  $a, b, c \in R$  and  $c$  be a common upper bound of  $a$  and  $b$ . Then  $a = C(a)c$  and  $b = C(b)c$ . By Lemma 2.4,  $a^*ra = a^*rc$ ,  $b^*rb = b^*rc$  for all  $r \in R$ . Also,  $b^*rb = c^*rb$  for all  $r \in R$ .

(1): Since  $a = C(a)c$  and  $b = C(b)c$ , we have  $aC(b) = C(a)cC(b) = bC(a)$ .  
 (2): Let  $r \in R$ . Then  $a^*rb = a^*rC(b)c = C(b)a^*rc = C(b)a^*ra$ . Also,  $a^*rb = (C(a)c)^*rb = C(a)c^*rb = C(a)b^*rb$ . Consequently,  $a^*rb = C(a)b^*rb = C(b)a^*ra$  for all  $r \in R$ . In particular for  $r = 1$ , we have  $a^*b = C(b)a^*a$ . Therefore  $(a^*b)^* = C(b)a^*a = a^*b$ . Thus  $a^*b$  is self adjoint.

(3): The proof is similar to the proof of part (1).

(4): To prove  $a \wedge b = aC(b)$ , first we prove that  $aC(b)$  is a common lower bound of  $a$  and  $b$ . By Remark 2.1,  $C(aC(b))a = C(a)C(b)a = aC(b)$ . This implies  $aC(b) \leq a$ . Similarly,  $bC(a) \leq b$ . By part (1), we get  $aC(b) \leq b$ . Let  $d \in R$  be such that  $d \leq a$  and  $d \leq b$ . Then  $d = C(d)a = C(d)b$  and hence  $dC(b) = C(d)b$ . Further,  $C(d)aC(b) = dC(b) = C(d)b = d$ . Therefore  $d \leq aC(b)$ . Thus  $a \wedge b = aC(b) = bC(a)$ .

(5): By (1) and (4),  $C(a)(a+b-a \wedge b) = C(a)(a+b-aC(b)) = aC(a) + bC(a) - aC(a)C(b) = a + bC(a) - aC(b) = a$ . This yields  $a \leq (a + b - a \wedge b)$ . Similarly,  $b \leq (a + b - a \wedge b)$ . Let  $d \in R$  be such that  $a \leq d$  and  $b \leq d$ . Then  $a = C(a)d$  and  $b = C(b)d$ . Let  $r \in R$ . By using part (2),  $(a + b - a \wedge b)^*r(a + b - a \wedge b) = (a^* + b^* - a^*C(b))r(a + b - aC(b)) = a^*ra + a^*rb - a^*raC(b) + b^*ra + b^*rb - b^*raC(b) - a^*raC(b) - a^*rbC(b) + a^*raC(b) = a^*ra + a^*rb - a^*rb + b^*ra + b^*rb - C(b)b^*ra - a^*rb - a^*raC(b) + a^*raC(b) = a^*ra + b^*ra + b^*rb - b^*ra - a^*rb = a^*rdC(a) + b^*rdC(b) - a^*rdC(b) = a^*rd + b^*rd - a^*rdC(b) = (a^* + b^* - a^*C(b))rd = (a + b - aC(b))^*rd = (a + b - a \wedge b)^*rd$ . Thus, by Lemma 2.4,  $(a + b - a \wedge b) \leq d$ . Therefore  $a \vee b = a + b - a \wedge b$ .  $\square$

As an immediate consequence of above theorem we have following corollaries.

**Corollary 3.6.** *Let  $R$  be a p.q.-Baer \*-ring. Then  $R$  is a pseudo lattice with respect to Conrad's partial order.*

**Corollary 3.7.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . If  $a \vee b$  exists in  $R$  then  $a \vee b = a + b(1 - C(a)) = b + a(1 - C(b))$ .*

By Theorem 3.5(1), in a p.q.-Baer \*-ring  $R$ , if  $a, b \in R$  have a common upper bound then  $aC(b) = bC(a)$ . In the following lemma, we prove that the converse of this statement is also true.

**Lemma 3.8.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . If  $aC(b) = bC(a)$  then  $a, b$  have a common upper bound.*

*Proof.* Let  $a, b \in R$  be such that  $aC(b) = bC(a)$ . We prove that  $a + b - aC(b)$  is a common upper bound of  $a$  and  $b$ . We have  $C(a)(a + b - aC(b)) = a + C(a)b - aC(b) = a$ . Also,  $C(b)(a + b - aC(b)) = aC(b) + b - aC(b) = b$ . Therefore  $a \leq a + b - aC(b)$  and  $b \leq a + b - aC(b)$ , as required.  $\square$

**Corollary 3.9.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . Then,  $aC(b) = bC(a)$  if and only if  $a \vee b = a + b - a \wedge b$*

The following theorem, characterises p.q.-Baer \*-rings which form lattices with Conrad's partial order.

**Theorem 3.10.** *Let  $R$  be a p.q.-Baer \*-ring. Then  $R$  is a lattice with respect to Conrad's partial order if and only if  $aC(b) = bC(a)$  for all  $a, b \in R$ .*

*Proof.* The proof follows from Theorem 3.5 and Lemma 3.8.  $\square$

We conclude this section with positive answer to Problem 2, when  $R$  is a p.q.-Baer \*-ring with Conrad's partial order.

**Theorem 3.11.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b, c \in R$ . If  $a \leq c$ ,  $b \leq c$ ,  $aR \cap bR = \{0\}$  then  $a + b \leq c$ .*

*Proof.* Let  $a, b, c \in R$ ,  $a \leq c$ ,  $b \leq c$  and  $aR \cap bR = \{0\}$ . Then, by Theorem 3.5,  $aC(b) = bC(a)$ . This implies  $aC(b) \in aR \cap bR$  and hence  $aC(b) = 0$ . Again, by using Theorem 3.5,  $a \vee b = a + b$ . Thus  $a + b \leq c$ .  $\square$

#### 4. ORTHOGONALITY RELATION ON P.Q.-BAER \*-RINGS

In this section, we prove that the initial segments of an arbitrary p.q.-Baer \*-rings with Conrad's partial order are orthomodular posets.

We recall the following definitions from [7].

A binary relation  $\perp$  on a poset  $(P, \leq, 0)$ , where  $0$  is the least element of the poset, is called an *orthogonality relation* (for the order  $\leq$ ) if for all  $x, y, z \in P$ ,

- (1) if  $x \perp y$ , then  $y \perp x$ ;
- (2) if  $x \leq y$  and  $y \perp z$ , then  $x \perp z$ ; and
- (3)  $0 \perp x$ .

A poset with orthogonality  $(P, \leq, \perp, 0)$  is called *quasi-orthomodular* if for all  $x, y \in P$ ,

- (4) if  $x \perp y$ , then  $x \vee y$  exists;
- (5) if  $x \leq y$ , then  $y = x \vee z$  for some  $z \in P$  with  $x \perp z$ ;
- (6) if  $x \perp y$ ,  $x \perp z$  and  $y \leq x \vee z$ , then  $y \leq z$ .

A poset  $(P, \leq, 0, 1)$  (where  $0$  is the least and  $1$  is the greatest element) is called an *orthocomplemented poset* if there is an operation  $^\perp : P \rightarrow P$  such that for all  $a, b \in P$ ,

- (1)  $a \wedge a^\perp$  and  $a \vee a^\perp$  exist, and  $a \wedge a^\perp = 0$  and  $a \vee a^\perp = 1$ ;
- (2)  $(a^\perp)^\perp = a$ ;
- (3) if  $a \leq b$ , then  $b^\perp \leq a^\perp$ .

The operation  $^\perp$  is called an orthocomplementation. In an orthocomplemented poset, we define the relation  $\perp$  by  $a \perp b$  if and only if  $a \leq b^\perp$ . This is an orthogonality relation. An orthocomplemented poset  $(P, \leq, ^\perp, 0, 1)$  is called orthomodular if for all  $a, b \in P$ ,

- (1) if  $a \perp b$ , then  $a \vee b$  exist;
- (2) if  $a \leq b$ , then there exists an element  $c \in P$  such that  $c \leq a^\perp$  and  $b = a \vee c$ .

Between orthomodularity and quasi-orthomodularity, the following connection holds.

**Theorem 4.1** ([7]). *In a quasi-orthomodular poset  $(P, \leq, ^\perp)$ , all initial segments  $[0, p] = \{a \in P \mid a \leq p\}$  are orthomodular for some orthogonality  $\perp_p$  on  $([0, p], \leq)$ . Furthermore, if  $\perp_p$  is the orthogonality of the initial segment  $[0, p]$ , then for all  $a, b \in [0, p]$ ,  $a \perp_p b$  if and only if  $a \perp b$ . Moreover, if  $x \perp_p y$  and  $x, y \leq q$ , then  $x \perp_q y$ .*

By using above theorem, we prove that the initial segments of p.q.-Baer \*-rings with Conrad's partial order are orthomodular posets, for that we prove the following sequence of theorems and lemmas.

**Theorem 4.2.** *The relation  $\perp$  is an orthogonality relation on a p.q.-Baer \*-ring.*

*Proof.* Let  $R$  be a p.q.-Baer \*-ring. By definition of orthogonal elements, it is clear that for any  $x, y \in R$ , if  $x \perp y$  then  $y \perp x$ . Suppose  $a \leq b$  and  $b \perp c$ . Then  $a = C(a)b$  and  $C(b)C(c) = 0$ . By Lemma 2.5,  $C(a)C(c) = C(a)C(b)C(c) = 0$  and hence  $a \perp c$ . Further,  $C(0) = 0$ , therefore  $C(0)C(x) = 0$  for any  $x \in R$ . Consequently,  $0 \perp x$  for any  $x \in R$ . Thus the relation  $\perp$  is an orthogonality relation.  $\square$

**Lemma 4.3.** *Let  $R$  be a p.q.-Baer \*-ring. If  $a$  and  $b$  are orthogonal elements of  $R$ , then  $a$  and  $b$  have a common upper bound.*

*Proof.* Let  $a, b \in R$  be such that  $a \perp b$ . Then  $C(a)C(b) = 0$ . This implies  $aC(b) = C(a)b = 0$ . Therefore by Lemma 3.8,  $a$  and  $b$  have a common upper bound.  $\square$

In the following theorem, we prove that orthogonal elements of a p.q.-Baer \*-ring possess the join and the meet.

**Theorem 4.4.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  be orthogonal elements. Then  $a \wedge b$ ,  $a \vee b$  exist and  $a \wedge b = 0$ ,  $a \vee b = a + b$ .*

*Proof.* Let  $a, b \in R$  be orthogonal elements. Then by Lemma 4.3,  $a$  and  $b$  have a common upper bound. By Theorem 3.5,  $a \wedge b = aC(b)$  and  $a \vee b = a + b - aC(b)$ . Since  $a$  and  $b$  are orthogonal elements, we have  $aC(b) = 0$ . Therefore  $a \wedge b = 0$  and  $a \vee b = a + b$ .  $\square$

The following lemma leads to the orthomodularity condition in a poset.

**Lemma 4.5.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . If  $a \leq b$  then there exists  $c \in R$  such that  $a \perp c$  and  $b = a + c$ .*

*Proof.* Let  $a, b \in R$  and  $a \leq b$ . Then  $a = C(a)b$  and hence  $C(a) = C(a)C(b)$ . Let  $c = b - a$ . By Lemma 2.5,  $C(a)C(c) = C(a)C(b - a) = C(a)(C(b) - C(a)) = C(a)C(b) - C(a) = 0$ . Therefore  $a \perp c$ .  $\square$

**Lemma 4.6.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b, c \in R$ . If  $a \perp b$ ,  $a \perp c$  and  $b \leq a \vee c$ , then  $b \leq c$ .*

*Proof.* Let  $a, b, c \in R$  be such that  $a \perp b$ ,  $a \perp c$  and  $b \leq a \vee c$ . Then  $C(a)C(b) = C(a)C(c) = 0$  and  $b = C(b)(a \vee c)$ . By Theorem 4.4,  $b = C(b)(a + c) = C(b)a + C(b)c = C(b)c$ . Thus  $b \leq c$ .  $\square$

**Theorem 4.7.** *A p.q.-Baer \*-ring with the order  $\leq$  and the orthogonality  $\perp$  is a quasi-orthomodular poset.*

*Proof.* The proof follows from Theorem 4.4 and Lemmas 4.5 and 4.6.  $\square$

**Theorem 4.8.** *In a p.q.-Baer \*-ring  $R$ , the initial segments  $[0, m] = \{a \in R \mid a \leq m\}$  are orthomodular posets. Furthermore, if  $\perp_m$  is the local orthogonality of the initial segment  $[0, m]$ , then for all  $a, b \in [0, m]$ ,  $a \perp_m b$  if and only if  $a \perp b$ . Moreover, if  $a \perp_m b$  and  $a, b \leq n$ , then  $a \perp_n b$ .*

*Proof.* The proof follows from Theorems 4.1 and 4.7.  $\square$

**Acknowledgment:** The first author gratefully acknowledges the University Grant Commission, New Delhi, India for the award of Teachers Fellowship under the faculty development program, during the  $XII^{th}$  plan period (2012-2017).

## REFERENCES

- [1] A. Abian, *Direct product decomposition of commutative semisimple rings*, Proc. Amer. Math. Soc. **24** (1970), 502-507.
- [2] J.K. Baksalary and S.K. Mitra, *Left-star and right-star partial ordering*, Linear Algebra Appl. **149** (1991), 73-89.
- [3] S.K. Berberian, *Baer \*-Rings*, Grundlehren Math. Wiss. Band 195. Vol. **296**, Berlin: Springer (1972).
- [4] G.F. Birkenmeier, J.K. Park, S.R. Tariq, *Extensions of Rings and Modules*, New York: Birkhäuser (2013).
- [5] B. Blackwood, S.K. Jain, K.M. Pasad and A.K. Srivastava, *Shorted Operators Relative to a Partial Order in a Regular Ring*, Comm. Algebra **37** (11) (2009), 4141-4152.
- [6] W.D. Burgess and R. Raphael, *On Conrad's partial order relation on semiprime rings and on semigroups*, Semigroup Forum **16** (1978), 133-140.
- [7] J. Ciriulis, *Quasi-orthomodular posets and weak BCK-algebras*, Order **31** (2014), 403-419.
- [8] P.F. Conrad, *The hulls of semiprime rings*, Austral. Math. Soc. **12** (1975), 311-314.
- [9] G. Dolinar and J. Marovt, *Star partial order on  $B(H)$* , Linear Algebra Appl. **434** (2011), 319-326.
- [10] G. Dolinar, B. Kuzma and J. Marovt, *A note on partial orders of Hartwig, Mitsch, and Šemrl*, Appl. Math. and Comp. **270** (2015), 711-713.
- [11] M.P. Drazin, *Natural structure on semigroup with involution*, Bull. Amer. Math. Soc. **84** (1) (1978), 139-141.
- [12] R.E. Hartwig, *How to partially order regular elements*, Math. Japon. **25** (1) (1980), 1-13.
- [13] R.E. Hartwig, *Pseudo lattice properties of the star-orthogonal partial ordering for star-regular rings.*, Proc. Amer. Math. Soc. **77** (3) (1979), 299-303.
- [14] M.F. Janowitz, *On the \*-order for Rickart \*-rings*, Algebra Universalis **16** (1983), 360-369.
- [15] I. Kaplansky, *Rings of operators*, W. A. Benjamin, Inc. New York-Amsterdam (1968).
- [16] A. Khairnar and B.N. Waphare, *Unification of Weakly p.q.-Baer \*-rings*, (Communicated).
- [17] I. Krěmère, *Left-star order structure of Rickart \*-ring*, Linear Multilinear Algebra **64** (3) (2016), 341-352.
- [18] S. Maeda, *On the lattice of projections of a Baer \*-ring*, J. Sci. Hiroshima Univ. Ser. A **22** (1958), 75-88.
- [19] P. Šemrl, *Automorphisms of  $B(H)$  with respect to minus partial order*, J. Math. Anal. Appl. **369** (2010), 205-213.

